Transverse Kinetic Stability*

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Transverse Kinetic Stability: Outline

Transverse Kinetic Stability: Detailed Outline

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S1: Overview: Machine Operating Points

Good transport of a single component beam with intense space-charge described by a Vlasov-Poisson type model requires:

Lowest Order:

1. Stable single-particle centroid

Next Order:

2. Stable rms envelope

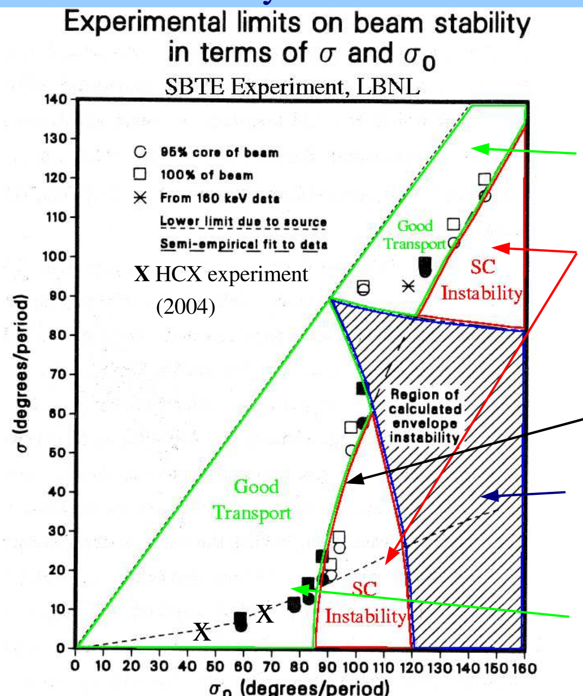
Higher Order:

3. "Stable" Vlasov description

Transport of a relatively smooth initial beam distribution can fail or become "unstable" within the Vlasov model for several reasons:

- Collective modes internal to beam become unstable and grow
  - Large amplitudes can lead to statistical (rms) beam emittance growth
- Excessive halo generated
  - Increased statistical beam emittance and particle losses
- Combined processes above

Transport limits in periodic (FODO) quadrupole lattices that result from higher order processes have been measured in the SBTE experiment. These results have only a limited theoretical understanding in 20+ years



Limits defined with respect to reasonable (smooth) initial distributions

Low Space-Charge Intensity Transport

Emittance Blow Up (Unexplained)
-- Not Practical for Applications

Empirical fit to higher-order stability boundary

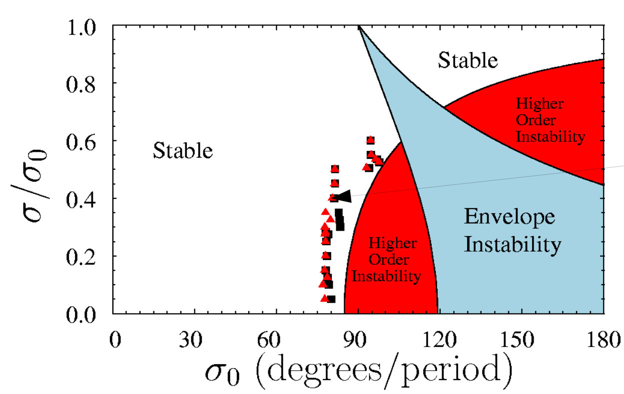
$$\sigma_0^2 - \sigma^2 = \frac{1}{2}(120°)^2$$

Envelope Instability
– Not Practical for Applications

High Space-Charge Intensity Transport
- Valid for Practical Applications

[M.G. Tiefenback, Ph.D Thesis, UC Berkeley (1986)]

Summary of beam stability with intense space-charge in a quadrupole transport lattice: centroid, envelope, and theory boundary based on higher order emittance growth/particle losses



Theory stability boundary points

New theory analyzes processes relating to AG transport limits without equilibria

- Suggests near core, chaotic halo resonances can drive strong emittance growth and particle losses
- Results checked with fully self-consistent simulations

Analogous results (with less "instability") exist for solenoidal transport

[Lund and Chawla, NIMA 561 203 (2006)]

S2: Overview:
Collective Modes and Transverse Kinetic Stability

In discussion of transverse beam physics we have focused on:

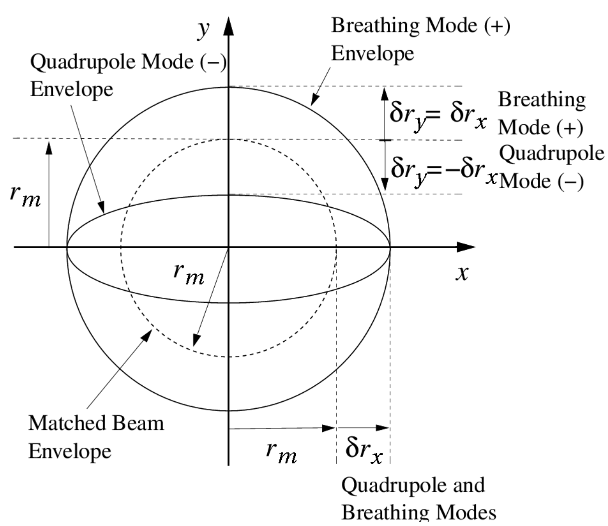
Equilibrium

- Used to estimate balance of space-charge and focusing forces
  - KV model for periodic focusing
  - Continuous focusing equilibria for qualitative guide on space-charge effects such as Debye screening and nonlinear equilibrium self-field effects

Centroid/Envelope Modes and Stability

- Lowest order collective oscillations of the beam
  - Analyzed assuming fixed internal form of the distribution
- Model only exactly correct for KV equilibrium distribution
  - Should hold in a leading-order sense for a wide variety of real beams
- Predictions of instability regions are well verified by experiment
  - Significantly restricts allowed system parameters for periodic focusing lattices

Example – Envelope Modes on a Round, Continuously Focused Beam

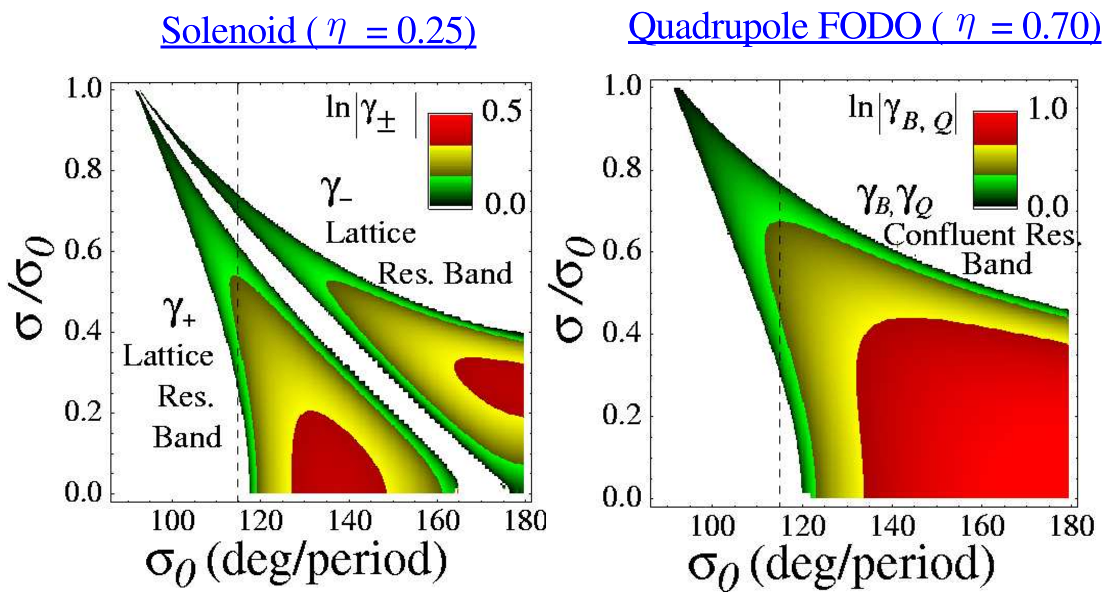


The analog of these modes in a periodic focusing lattice can be destabilized

- Constrains system parameters to avoid band (parametric) regions of instability

Reminder (lecture on *Centroid and Envelope Descriptions of Beams*): Instability bands of the KV envelope equation are well understood in periodic focusing channels

Envelope Mode Instability Growth Rates

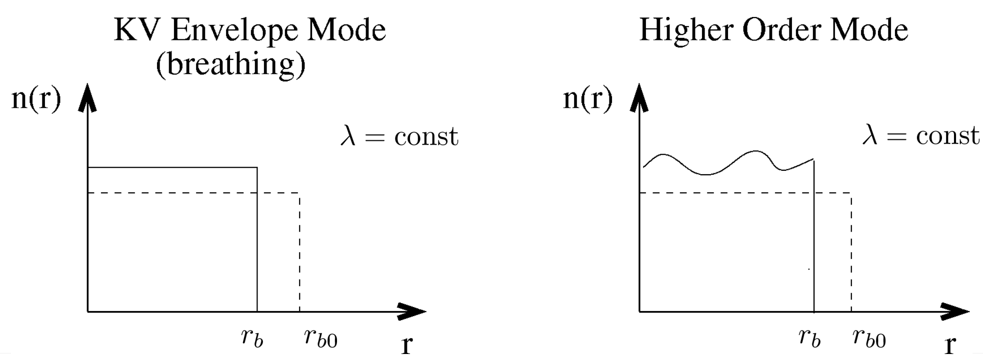


[S.M. Lund and B. Bukh, PRSTAB 024801 (2004)]

More instabilities are possible than can be described by statistical (moment/envelope) equations. Look at a more complete, Vlasov based kinetic theory including self-consistent space-charge:

Higher-order Collective (internal) Mode Stability

- Perturbations will generally drive nonlinear space-charge forces
- Evolution of such perturbations can change the beam rms emittance
- Many possible internal modes of oscillation should be possible
  - Frequencies can differ significantly from envelope modes
  - Creates more possibilities for resonant exchanges with a periodic focusing lattice and various beam characteristic responses opening many possibilities for system destabilization



Plasma physics approach to beam physics:

Resolve:

$$\boxed{f(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) = f_\perp(\{C_i\}) + \delta f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)}$$

equilibrium perturbation $f_\perp \gg \delta f_\perp$

and carry out equilibrium + stability analysis

Comments:

- Attraction is to parallel the impressive successes of plasma physics
  - Gain insight into preferred state of nature
- Beams are born off a source and may not be close to an equilibrium condition
  - Appropriate single particle constants of the motion unknown for periodic focusing lattices other than the KV distribution
- Intense beam self-fields and finite radial extent vastly complicate equilibrium description and analysis of perturbations

Review: Vlasov Model: Transverse Vlasov model for a coasting, single species beam with electrostatic self-fields propagating in an applied focusing lattice:

$\mathbf{x}_\perp, \ \mathbf{x}'_\perp$	transverse particle coordinate, angle		
$q, \ m$	charge, mass	$f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$	single particle distribution
$\gamma_b, \ \beta_b$	axial relativistic factors	$H_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$	single particle Hamiltonian

Vlasov Equation (see J.J. Barnard, Introductory Lectures):

$$\frac{d}{ds} f_\perp = \frac{\partial f_\perp}{\partial s} + \frac{d\mathbf{x}_\perp}{ds} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}_\perp} + \frac{d\mathbf{x}'_\perp}{ds} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0$$

Particle Equations of Motion:

$$\frac{d}{ds}\mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} \qquad \frac{d}{ds}\mathbf{x}'_\perp = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp}$$

Hamiltonian (see S.M. Lund, lectures on *Transverse Particle Equations of Motion*):

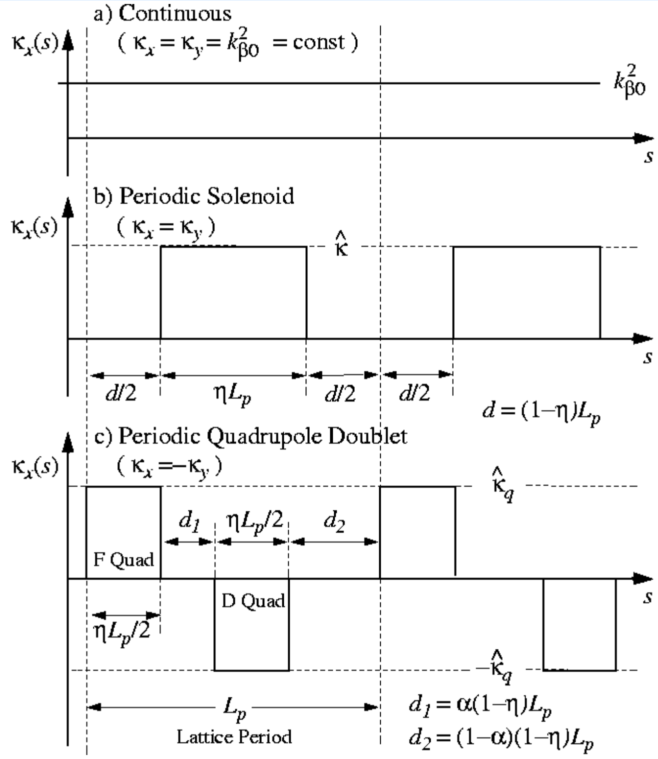
$$H_\perp = \frac{1}{2}\mathbf{x}'^{\,2}_\perp + \frac{1}{2}\kappa_x(s)x^2 + \frac{1}{2}\kappa_y(s)y^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

Poisson Equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{q}{\epsilon_0}\int d^2\mathbf{x}'_\perp \ f_\perp$$

+ boundary conditions on ϕ

Review: Focusing lattices, continuous and periodic (simple piecewise constant):



Lattice Period L_p

Occupancy η

$\eta \in [0, 1]$

Solenoid description carried out implicitly in Larmor frame [see Lund and Bukh, PRST- Accel. and Beams 7, 024801 (2004), Appendix A]

Syncopation Factor α

$$\alpha \in [0, \frac{1}{2}]$$

$$\alpha = \frac{1}{2} \implies FODO$$

Continuous Focusing: $\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const}$

$$\boxed{H_\perp = \frac{1}{2}\mathbf{x}_\perp'^{\,2} + \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi}$$

Solenoidal Focusing (in Larmor frame variables): $\kappa_x = \kappa_y = \kappa(s)$

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^{\,2} + \frac{1}{2}\kappa \mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

Quadrupole Focusing: $\kappa_x = -\kappa_y = \kappa_q(s)$

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^{\,2} + \frac{1}{2}\kappa_q x^2 - \frac{1}{2}\kappa_q y^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

We will concentrate on the continuous focusing model in these lectures

- Kinetic theory is notoriously complicated even in this (simple) case
- By analogy with envelope mode results expect that kinetic theory of periodic focusing systems to have more instabilities
- As in equilibrium analysis the continuous model can give simplified insight on a range of relevant kinetic stability considerations

S3: Linearized Vlasov Equation

Because of the complexity of kinetic theory, we will limit discussion to a simple continuous focusing model Vlasov-Poisson system for a coasting beam within a round pipe

$$\frac{df_\perp}{ds} = \left\{ \frac{\partial}{\partial s} + \mathbf{x}'_\perp \cdot \frac{\partial}{\partial \mathbf{x}_\perp} - \left(k_{\beta 0}^2 \mathbf{x}_\perp + \frac{q}{m\gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial \mathbf{x}_\perp} \right) \cdot \frac{\partial}{\partial \mathbf{x}'_\perp} \right\} f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) = 0$$

$$\nabla_\perp^2 \phi(\mathbf{x}_\perp, s) = -\frac{q}{\epsilon_0} \int d^2x'_\perp \; f_\perp(\mathbf{x}_\perp, \mathbf{x}_\perp, s)$$

$$\phi(|\mathbf{x}_\perp| = r_p, s) = \text{const}$$

Then expand the distribution and field as:

$$\begin{aligned} f_\perp &= \underbrace{f_0(H_0)}_{} + \delta f_\perp \\ \phi &= \phi_0 + \delta\phi \end{aligned}$$

equilibrium perturbation

At present, there is *no assumption* that the perturbations are small.

The equilibrium satisfies:

(see S.M. Lund, lectures on *Transverse Equilibrium Distributions*)

$$H_0 = \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3 \beta_b^2 c^2}\phi_0$$

$$f_0(H_0) = \quad \text{any non-negative function}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi_0}{\partial r} \right) = -\frac{q}{\epsilon_0} \int d^2x'_\perp \; f_0(H_0)$$

The unperturbed distribution must then satisfy the equilibrium Vlasov equation:

$$\left\{ \frac{\partial}{\partial s} + \mathbf{x}'_\perp \cdot \frac{\partial}{\partial \mathbf{x}_\perp} - \left(k_{\beta 0}^2 \mathbf{x}_\perp + \frac{q}{m\gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi_0}{\partial \mathbf{x}_\perp} \right) \cdot \frac{\partial}{\partial \mathbf{x}'_\perp} \right\} f_0(H_0) = 0$$

Because the Poisson equation is linear:

$$\nabla_\perp^2 \delta\phi(\mathbf{x}_\perp, s) = -\frac{q}{\epsilon_0} \int d^2x'_\perp \; \delta f_\perp(\mathbf{x}_\perp, \mathbf{x}_\perp, s)$$

$$\delta\phi(|\mathbf{x}_\perp| = r_p, s) = \text{const}$$

Insert the perturbations in Vlasov's equation and expand terms:

$$\overbrace{\left\{ \frac{\partial}{\partial s} + \mathbf{x}'_\perp \cdot \frac{\partial}{\partial \mathbf{x}_\perp} - \left(k_{\beta 0}^2 \mathbf{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi_0}{\partial \mathbf{x}_\perp} \right) \cdot \frac{\partial}{\partial \mathbf{x}'_\perp} \right\} f_0(H_0)}^{0}$$ equilibrium term

$$+ \left\{ \frac{\partial}{\partial s} + \mathbf{x}'_\perp \cdot \frac{\partial}{\partial \mathbf{x}_\perp} - \left(k_{\beta 0}^2 \mathbf{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi_0}{\partial \mathbf{x}_\perp} \right) \cdot \frac{\partial}{\partial \mathbf{x}'_\perp} \right\} \delta f_\perp$$ equilibrium characteristics of perturbed distribution

$$= \underbrace{\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \delta\phi}{\partial \mathbf{x}_\perp} \cdot \frac{\partial}{\partial \mathbf{x}'_\perp} f_0(H_0)}_{\text{perturbed field linear correction term}} + \underbrace{\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \delta\phi}{\partial \mathbf{x}_\perp} \cdot \frac{\partial}{\partial \mathbf{x}'_\perp} \delta f_\perp}_{\text{nonlinear term}}$$

Take the perturbations to be small-amplitude:

$$\boxed{\begin{aligned} f_0(H_0) &\gg \delta f_\perp \\ \phi_0 &\gg \delta\phi \end{aligned}}$$

<--- follows automatically from distribution/Poisson eq

and neglect the nonlinear terms to obtain the linearized Vlasov-Poisson system:

$$\boxed{\begin{aligned} &\left\{ \frac{\partial}{\partial s} + \mathbf{x}'_\perp \cdot \frac{\partial}{\partial \mathbf{x}_\perp} - \left(k_{\beta 0}^2 \mathbf{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi_0}{\partial \mathbf{x}_\perp} \right) \cdot \frac{\partial}{\partial \mathbf{x}'_\perp} \right\} \delta f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) \\ &\qquad = \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \delta\phi(\mathbf{x}_\perp, s)}{\partial \mathbf{x}_\perp} \cdot \frac{\partial}{\partial \mathbf{x}'_\perp} f_0(H_0) \\ &\nabla_\perp^2 \delta\phi(\mathbf{x}_\perp, s) = -\frac{q}{\epsilon_0}\int d^2x'_\perp \; \delta f_\perp(\mathbf{x}_\perp, \mathbf{x}_\perp, s) \qquad \delta\phi(|\mathbf{x}_\perp| = r_p, s) = \text{const} \end{aligned}}$$

Solution of the Linearized Vlasov Equation, the method of characteristics

The linearized Vlasov equation is a integral-partial differential equation system

- Highly nontrivial to solve
- Method of characteristics can be employed to simplify analysis due to the structure of the equation

Note that the equilibrium Vlasov equation is:

$$\left\{ \frac{\partial}{\partial s} + \mathbf{x}'_\perp \cdot \frac{\partial}{\partial \mathbf{x}_\perp} - \left(k_{\beta 0}^2 \mathbf{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi_0}{\partial \mathbf{x}_\perp} \right) \cdot \frac{\partial}{\partial \mathbf{x}'_\perp} \right\} f_0 = 0$$

$$\left.\frac{d}{ds}\right|_{\text{eq. orbit}} f_0 = 0$$

Interpret:

$$\left\{ \frac{\partial}{\partial s} + \mathbf{x}'_\perp \cdot \frac{\partial}{\partial \mathbf{x}_\perp} - \left(k_{\beta 0}^2 \mathbf{x}_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi_0}{\partial \mathbf{x}_\perp} \right) \cdot \frac{\partial}{\partial \mathbf{x}'_\perp} \right\} = \left.\frac{d}{ds}\right|_{\text{eq. orbit}}$$

as a total derivative evaluated along an equilibrium particle orbit. This suggests employing the method of characteristics.

Method of Characteristics:

Equilibrium orbit:

$$\boxed{\begin{aligned} &\frac{d}{d\tilde{s}}\mathbf{x}_\perp(\tilde{s}) = \mathbf{x}'_\perp(\tilde{s}) \\ &\frac{d}{d\tilde{s}}\mathbf{x}'_\perp(\tilde{s}) = -k_{\beta 0}^2\mathbf{x}_\perp(\tilde{s}) - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi_0(\mathbf{x}_\perp(\tilde{s}))}{\partial\mathbf{x}_\perp(\tilde{s})} \end{aligned}}$$

"Initial" conditions of characteristic orbit:

$$\boxed{\begin{aligned} \mathbf{x}_\perp(\tilde{s} = s) &= \mathbf{x}_\perp \\ \mathbf{x}'_\perp(\tilde{s} = s) &= \mathbf{x}'_\perp \end{aligned}}$$

Then the linearized Vlasov equation can be expressed as:

$$\frac{d}{d\tilde{s}}\delta f_\perp(\mathbf{x}_\perp(\tilde{s}), \mathbf{x}'_\perp(\tilde{s}), \tilde{s}) = \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\delta\phi(\mathbf{x}_\perp(\tilde{s}))}{\partial\mathbf{x}_\perp(\tilde{s})}\cdot\frac{\partial}{\partial\mathbf{x}'_\perp}f_0(H_0(\mathbf{x}_\perp(\tilde{s}), \mathbf{x}'_\perp(\tilde{s})))$$

This is a total derivative and can be integrated:

- To analyze instabilities assume growing perturbations that grow in s
- Neglect initial conditions at $\tilde{s} \to -\infty$ and integrate

$$\boxed{\begin{gathered} \delta f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) = \frac{q}{m\gamma_b^3\beta_b^2c^2}\int_{-\infty}^{s} d\tilde{s}\,\frac{\partial\delta\phi(\mathbf{x}_\perp(\tilde{s}))}{\partial\mathbf{x}_\perp(\tilde{s})}\cdot\frac{\partial}{\partial\mathbf{x}'_\perp}f_0(H_0(\mathbf{x}_\perp(\tilde{s}), \mathbf{x}'_\perp(\tilde{s}))) \\ \nabla_\perp^2\delta\phi(\mathbf{x}_\perp, s) = -\frac{q}{\epsilon_0}\int d^2x'_\perp\;\delta f_\perp(\mathbf{x}_\perp, \mathbf{x}_\perp, s) \\ \delta\phi(|\mathbf{x}_\perp| = r_p, s) = \text{const} \end{gathered}}$$

Gives the self-consistent evolution of the perturbations

- Similar statement for nonlinear perturbations (Homework problem)

Effectively restates the Poisson equation as a differential-integral equation that is solved to understand the evolution of perturbations

- Simpler to work with but still *very* complicated

S4: Collective Modes on a KV Equilibrium Beam

Unfortunately, calculation of normal modes is generally complicated even in continuous focusing. Nevertheless, the normal modes of the KV distribution can be analytically calculated and give insight on the expected collective response of a beam with intense space-charge.

Review: Continuous Focusing KV Equilibrium

$$\boxed{f_\perp(H_\perp) = \frac{\hat{n}}{2\pi}\delta\left(H_\perp - \frac{\varepsilon^2}{2r_b^2}\right)}$$

$$r_b = \left(\frac{Q + \sqrt{4k_{\beta0}^2\varepsilon^2 + Q^2}}{2k_{\beta0}^2}\right)^{1/2} = \text{const}$$

$k_{\beta0} =$ Undepressed betatron wavenumber

$r_b =$ Beam edge radius

$\hat{n} =$ Beam number density

$Q =$ Dimensionless perveance

$\varepsilon =$ rms edge emittance

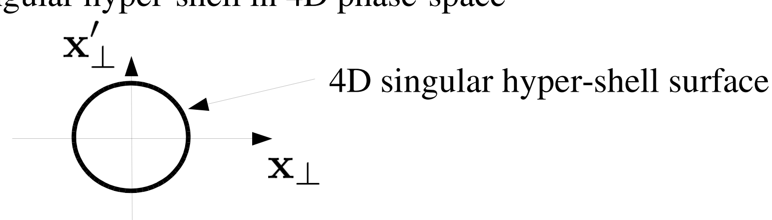
Further comments on the KV equilibrium: Distribution Structure

Equilibrium distribution:

$$\boxed{f_\perp \sim \delta[\text{Courant-Snyder invariants}]}$$

Forms a highly singular hyper-shell in 4D phase-space

Schematic:



- Singular distribution has large "Free-Energy" to drive many instabilities
  - Low order envelope modes are physical and highly important (see lectures on *Centroid and Envelope Descriptions of Beams*)
- Perturbative analysis shows strong collective instabilities
  - Hofmann, Laslett, Smith, and Haber, Part. Accel. **13**, 145 (1983)
  - Higher order instabilities (collective modes) have unphysical aspects due to (delta-function) structure of distribution and must be applied with care (see lectures on *Kinetic Stability of Beams*)
  - Instabilities can cause problems if the KV distribution is employed as an initial beam state in self-consistent simulations

A full kinetic stability analysis of the KV equilibrium distribution is complicated and uncovers many strong instabilities

[I. Hofmann, J.L. Laslett, L. Smith, and I. Haber, Particle Accel. 13, 145 (1983); R. Gluckstern, Proc. 1970 Proton Linac Conf., Batavia 811 (1971)]

Expand Vlasov's equation to linear order with:

$$\boxed{f_\perp \rightarrow f_\perp(\text{ C.S. Invariant}) \;+\; \delta f_\perp}$$

$f_\perp(\text{C.S. Invariant})$ = equilibrium

$\delta f_\perp$ = perturbation

Solve the Poisson equation:

$$\nabla_\perp^2 \delta\phi = -\frac{q}{\epsilon_0}\int d^2x' \; \delta f_\perp$$

using truncated polynomials for $\delta\phi$ internal to the beam to represent a "normal mode"

$$\delta\phi = \sum_{m=0}^{n} A_m^{(0)}(s) x^{n-m} y^m + \sum_{m=0}^{n-2} A_m^{(1)}(s) x^{n-m-2} y^m + \cdots$$

$n = 2, 3, 4, \cdots$ order or mode

m can be restricted to <u>even</u> or <u>odd</u> terms

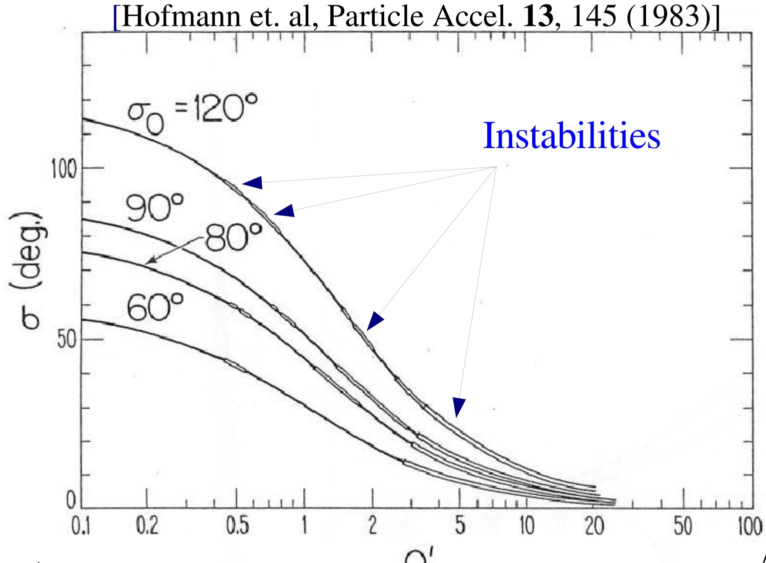
- Truncated polynomials can meet all boundary conditions
- Eigenvalues of a Floquet form transfer matrix analyzed for stability properties
  - Lowest order results reproduce KV envelope instabilities
  - Higher order results manifest many strong instabilities

Higher order kinetic instabilities of the KV equilibrium are strong and cover a wide parameter range for periodic focusing lattices

<u>Example: FODO Quadrupole Stability</u>

4th order even mode

[Hofmann et. al, Particle Accel. **13**, 145 (1983)]



$\sigma/\sigma_0 \rightarrow 1$ (space-charge parameter) $\sigma/\sigma_0 \rightarrow 0$

(undepressed) --> increasing space-charge --> (fully depressed)

The continuous focusing limit can be analyzed to better understand properties of internal modes on a KV beam (1)

[S. Lund and R. Davidson, Physics of Plasmas **5**, 3028 (1998): see Appendix B, C]

Continuous focusing, symmetric beam:

$$\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}$$

$$\varepsilon_x = \varepsilon_y \equiv \varepsilon$$

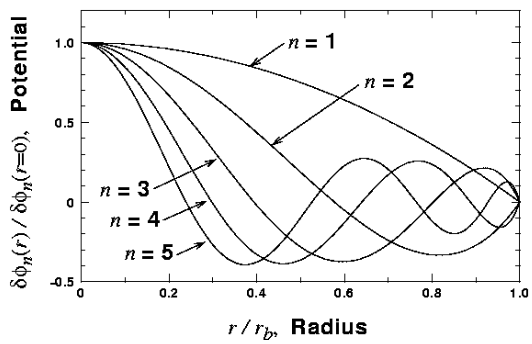
$$r_x = r_y \equiv r_b$$

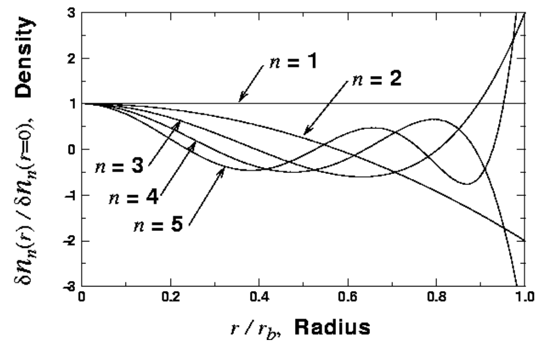
Mode eigenfunction (2 n "order"):

$$\delta\phi_n = \begin{cases} \frac{A_n}{2}\left[P_{n-1}\left(1-2\frac{r^2}{r_b^2}\right)+P_n\left(1-2\frac{r^2}{r_b^2}\right)\right], & 0 \le r \le r_b \\ 0, & r_b < r \end{cases}$$

$A_n = \text{const}$ $\quad$ $P_n(x) =$ n[th] order Legendre polynomial

Potential $\qquad$ Density $\left(\ \delta n_n = \epsilon_0 \nabla_\perp^2 \delta\phi_n / q\ \right)$





The continuous focusing limit can be analyzed to better understand properties of internal modes on a KV beam (2)

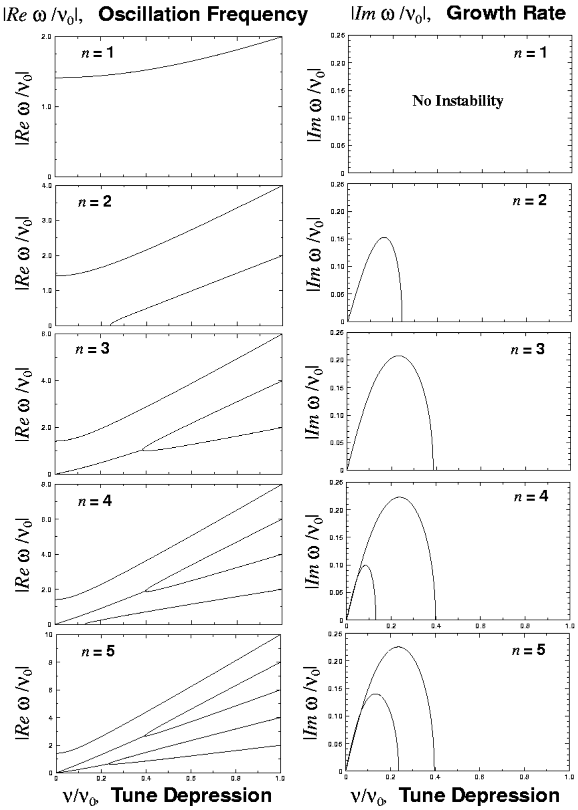
Mode dispersion relation for e^{-iks} variations:

$$2n + \frac{1-\sigma/\sigma_0}{(\sigma/\sigma_0)^2}\left[B_{n-1}\left(\frac{k/k_{\beta 0}}{\sigma/\sigma_0}\right) - B_n\left(\frac{k/k_{\beta 0}}{\sigma/\sigma_0}\right)\right] = 0$$

where: $$B_n(\alpha) \equiv \begin{cases} 1, & n = 0 \\ \frac{(\alpha/2)^2-0^2}{(\alpha/2)^2-1^2}\frac{(\alpha/2)^2-2^2}{(\alpha/2)^2-3^2}\cdots\frac{(\alpha/2)^2-(n-1)^2}{(\alpha/2)^2-n^2} & n = 1,3,5,\cdots \\ \frac{(\alpha/2)^2-1^2}{(\alpha/2)^2-2^2}\frac{(\alpha/2)^2-3^3}{(\alpha/2)^2-4^2}\cdots\frac{(\alpha/2)^2-(n-1)^2}{(\alpha/2)^2-n^2} & n = 2,4,6,\cdots \end{cases}$$

- Eigenfunction structure suggestive of wave perturbations often observed internal to the beam in simulations for a variety of beam distributions
- n distinct branches for 2n order (real coefficient) polynomial dispersion relation
- Some range of σ/σ_0 will be unstable for all n > 1
  - Instability exists for some n for $\sigma/\sigma_0 < 0.3985$
  - Growth rates are strong

Continuous focusing limit dispersion relation results for KV beam stability



Notation Change:

$$k/k_{\beta 0} \equiv \omega/\nu_0$$

$$\sigma/\sigma_0 \equiv \nu/\nu_0$$

[S. Lund and R. Davidson, Physics of Plasmas **5**, 3028 (1998): see Appendix B, C]

For continuous focusing, fluid theory shows that at least some branches of the KV dispersion relation are physical

[S. Lund and R. Davidson, Physics of Plasmas **5**, 3028 (1998)]

Fluid theory:

- KV equilibrium distribution is reasonable in fluid theory
  - No singularities
  - Flat density and parabolic radial temperature profiles
- Theory truncated by assuming zero heat flow

Mode eigenfunctions:

Exactly the same as derived under kinetic theory!

Mode dispersion relation:

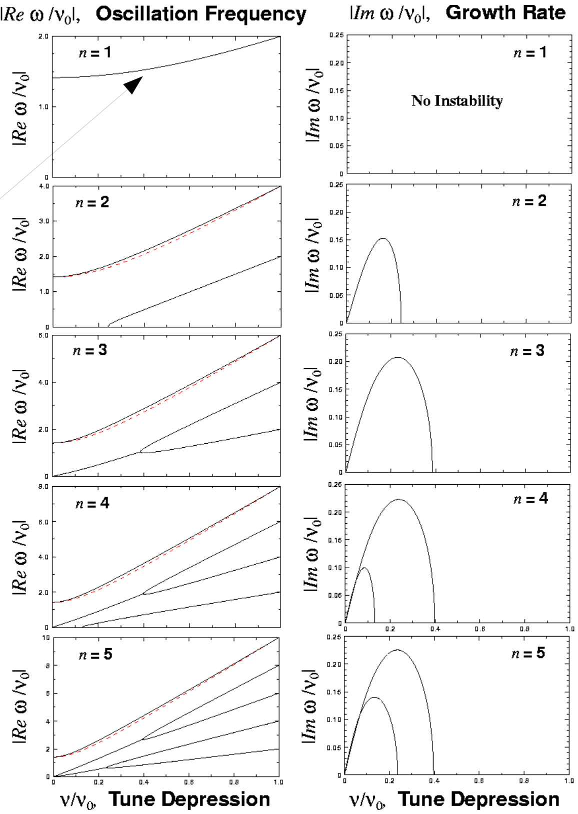
$$\boxed{\frac{k}{k_{\beta 0}} = \sqrt{2 + 2\left(\frac{\sigma}{\sigma_0}\right)^2 (2n^2 - 1)}}$$

- Single, stable branch
  - Agrees well with high frequency branch from kinetic theory

Results show that aspects of higher-order KV internal modes are physical!

Continuous focusing limit dispersion relation results for KV beam stability

n = 1, envelope mode curves overlap



Notation Change:

$$k/k_{\beta 0} \equiv \omega/\nu_0$$

$$\sigma/\sigma_0 \equiv \nu/\nu_0$$

Red: Fluid Theory (no instability)

Black: Kinetic Theory (unstable branches)

[S. Lund and R. Davidson, Physics of Plasmas **5**, 3028 (1998)]

S5: Global Conservation Constraints

Apply for any initial distribution, equilibrium or not.

- Strongly constrain nonlinear evolution of the system.
- Valid even with a beam pipe provided that particles are not lost from the system and that symmetries are respected.
- Useful to bound perturbations, but yields no information on evolution timescales.

1) Generalized Entropy

$$U_G = \int d^2x_\perp \int d^2x'_\perp \; G(f_\perp) \;\; = \text{const}$$

$$G(f_\perp) = \text{Any differentiable functions satisfying } G(f_\perp \to 0) = 0$$

- Applies to all Vlasov evolutions.

// Examples

Line-charge: $G(f_\perp) = q f_\perp \;\longrightarrow\; \boxed{\lambda = q \int d^2x \int d^2x' \; f_\perp \;\; = \text{const}}$

Entropy: $G(f_\perp) = -\dfrac{f_\perp}{A} \ln\left(\dfrac{f_\perp}{f_0}\right)$ $\quad A,\; f_0$ constants

$$\longrightarrow \boxed{\mathcal{S} = -\int \frac{d^2x}{A} \int d^2x' \; f_\perp \ln\left(\frac{f_\perp}{f_0}\right) \;\; = \text{const}}$$

//

2) Transverse Energy in continuous focusing

$$\boxed{U_{\mathcal{E}} = \int d^2x'_\perp \int d^2x_\perp \left\{\frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp\right\} f_\perp + \int d^2x_\perp \frac{\epsilon_0|\nabla_\perp\phi|^2}{2m\gamma_b^3\beta_b^2c^2} = \text{const}}$$

Here,

$$\int d^2x'_\perp \int d^2x_\perp \frac{1}{2}\mathbf{x}'^2_\perp f_\perp$$ ~ Kinetic Energy

$$\int d^2x'_\perp \int d^2x_\perp \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp f_\perp$$ ~ Potential Energy of applied focusing forces

$$\int d^2x_\perp \frac{\epsilon_0|\nabla_\perp\phi|^2}{2m\gamma_b^3\beta_b^2c^2}$$ ~ Self-Field Energy

- Does not hold when focusing forces vary in s
  - Can still be approximately valid for rms matched beams where energy will regularly pump into and out of the beam
- Self field energy term diverges in radially unbounded systems (no aperture)
  - Still useful if an appropriate infinite constant is subtracted (to regularize)

Comments on system energy form:

$$U_{\mathcal{E}} = \int d^2x'_\perp \int d^2x_\perp \left\{\frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp\right\} f_\perp + \int d^2x_\perp \frac{\epsilon_0|\nabla_\perp\phi|^2}{2m\gamma_b^3\beta_b^2c^2} = \text{const}$$

Analyze the energy term:

zero for grounded aperture in finite system

$$\int d^2x_\perp \frac{\epsilon_0|\nabla_\perp\phi|^2}{2} = \int d^2x_\perp \frac{1}{2}\nabla_\perp\cdot(\phi\nabla_\perp\phi) - \int d^2x_\perp \frac{1}{2}\phi\nabla^2_\perp\phi$$

or infinite constant in free space

Employ the Poisson equation:

$$\nabla^2_\perp\phi = -\frac{q}{\epsilon_0}\int d^2x'_\perp\, f_\perp$$

$$\longrightarrow \int d^2x_\perp \frac{\epsilon_0|\nabla_\perp\phi|^2}{2} = \int d^2x_\perp \int d^2x'_\perp \frac{q}{2\epsilon_0}\phi f_\perp$$

Giving:

$$\boxed{U_{\mathcal{E}} = \int d^2x'_\perp \int d^2x_\perp \left\{\frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp + \frac{1}{2}\frac{q\phi}{m\gamma_b^3\beta_b^2c^2}\right\} f_\perp = \text{const}}$$

symmetry factor

- Note the relation to the system Hamiltonian with a symmetry factor to not double count particle contributions

$$H_\perp = \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

Comments on self-field energy divergences:

In unbounded (free space) systems, far from the beam the field must look like a line charge:

$$-\frac{\partial \phi}{\partial r} \sim \frac{\lambda}{2\pi\epsilon_0 r} \qquad r > r_{\text{large}}$$

Resolve the total field energy into a finite (near) term and a divergent term:

$$\int d^2x_\perp \, \frac{\epsilon_0 |\nabla_\perp \phi|^2}{2} = \int_{r \le r_{\text{large}}} d^2x_\perp \, \frac{\epsilon_0 |\nabla_\perp \phi|^2}{2} + \frac{\lambda^2}{4\pi\epsilon_0} \int_{r_{\text{large}}}^{\infty} dr \frac{1}{r}$$

total | finite term | lograthmically divergent term

- This divergence can be subtracted out to thereby regularized the system energy
  - Renders energy constraint useful for application to equilibria in radially unbounded systems such as thermal equilibrium

3) Angular Momentum

$$\boxed{U_\theta = \int d^2x_\perp \int d^2x'_\perp \, (yx' - x'y) f_\perp \;\; = \text{const}}$$

- Focusing and beam pipe (if present) must be axisymmetric
  - Useful for solenoidal magnetic focusing
  - Does not apply to alternating gradient quadrupole focusing since such systems do not have the required axisymmetry

4) Axial Momentum

$$\boxed{U_z = \int d^2x_\perp \int d^2x'_\perp \, m\gamma_b\beta_b c \, f_\perp \;\; = \text{const}}$$

- Trivial here, but useful when models are generalized for coasting beams with axial momentum spread

Comments on applications of the global conservation constraints:

Global invariants strongly constrain the nonlinear evolution of the system
- Only evolutions consistent with Vlasov's equation are physical
- Constraints consistent with the model can bound kinematically accessible evolutions

Application of the invariants does not require (difficult to derive) normal mode descriptions
- But cannot, by itself, provide information on evolution timescales

Use of global constraints to bound perturbations has appeal since distributions in real machines may be far from an equilibrium. Used to:
- Derive sufficient conditions for stability
- Bound particle losses [O'Neil, Phys. Fluids **23**, 2216 (1980)]
- Bound changes of system moments (for example the rms emittance) under assumed relaxation processes
- Application does not require (difficult to derive) normal mode descriptions

S6: Kinetic Stability Theorem for continuous focusing equilibria

[Fowler, J. Math Phys. **4,** 559 (1963); Gardner, Phys. Fluids **6**, 839 (1963); R. Davidson, Physics of Nonneutral Plasmas, Addison-Wesley (1990)]

Resolve:

$$\boxed{\begin{aligned} &f_\perp = f_0(H_0) + \delta f_\perp \\ &\qquad f_0(H_0) = \quad \text{Equilibrium (subscript 0) distribution} \\ &\qquad \delta f_\perp = \quad \text{Perturbation about equilibrium} \end{aligned}}$$

Employ generalized entropy and transverse energy global constraints:

$$U_G = \int d^2x_\perp \int d^2x'_\perp \; G(f_\perp) \;\; = \text{const}$$

$$U_\mathcal{E} = \int d^2x'_\perp \int d^2x_\perp \left\{ \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp \right\} f_\perp \; + \; \int d^2x_\perp \; \frac{\epsilon_0 |\nabla_\perp \phi|^2}{2m\gamma_b^3\beta_b^2c^2} \;\; = \text{const}$$

Apply to equilibrium and full distribution to form an effective "free-energy":

$$\Delta U_G = U_G - U_{G0} = \text{const} \qquad\qquad \Delta U_\mathcal{E} = U_\mathcal{E} - U_{\mathcal{E}0} = \text{const}$$

$$\boxed{\begin{aligned} F &= \Delta U_\mathcal{E} - \Delta U_G \\ &= \int d^2x'_\perp \int d^2x_\perp \left\{ \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp \right\} \delta f_\perp \; + \; \int d^2x_\perp \; \frac{\epsilon_0 |\nabla_\perp \phi|^2}{2m\gamma_b^3\beta_b^2c^2} \\ &\qquad + \int d^2x_\perp \int d^2x'_\perp \; [G(f_\perp) - G(f_0)] \;\; = \text{const} \end{aligned}}$$

The perturbed potential satisfies:

$$\delta\phi \equiv \phi - \phi_0 \qquad \nabla_\perp^2 \delta\phi = -\frac{q}{\epsilon_0}\int d^2x'_\perp\ \delta f_\perp$$

Take $|\delta f_\perp| \ll f_0$ and Taylor expand to 2nd order

$$G(f_0 + \delta f_\perp) = G(f_0) + \frac{dG(f_0)}{df_0}\delta f_\perp + \frac{d^2G(f_0)}{df_0^2}\frac{(\delta f_\perp)^2}{2} + \Theta(\delta^3)$$

Without loss of generality, choose:

$$\frac{dG(f_0)}{df_0}\delta f_\perp = -H_0 = -\left(\frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp + \frac{q\phi}{m\gamma_b^2\beta_b^2c^2}\right)$$

Then

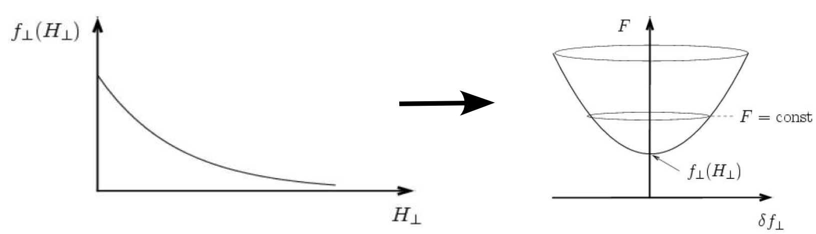
$$\frac{d^2G(f_0)}{df_0^2} = -\frac{\partial H_0}{\partial f_0} = \frac{-1}{\partial f_0(H_0)/\partial H_0}$$

Some algebra (few lines using partial integration) yields:

$$\boxed{F = \int d^2x_\perp \left\{\frac{\epsilon_0|\nabla_\perp\delta\phi|^2}{2m\gamma_b^3\beta_b^2c^2} - \int d^2x'_\perp \frac{(\delta f_\perp)^2}{\partial f_0(H_0)/\partial H_0}\right\} + \Theta(\delta^3) \ = \ \text{const}}$$

- If $\partial f_0(H_0)/\partial H_0 < 0$ then F is a sum of two positive definite terms and perturbations are bounded by F = const.

$$F = \int d^2x_\perp \left\{\frac{|\nabla_\perp\delta\phi|^2}{2m\gamma_b^3\beta_b^2c^2} - \int d^2x'_\perp \frac{(\delta f_\perp)^2}{\partial f_0(H_0)/\partial H_0}\right\} = \text{const}$$



Value of F set by initial perturbations and concavity bounds excursions

Drop zero subscripts in stability statement:

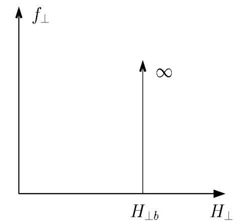
Kinetic Stability Theorem

> If $f_\perp(H_\perp)$ is a monotonic decreasing function of $H_\perp$ with $\partial f_\perp(H_\perp)/\partial H_\perp < 0$ then the equilibrium defined by $f_\perp(H_\perp)$ is stable to arbitrary small-amplitude perturbations.

- Is a sufficient condition for stability
  - Equilibria that violate may or may not be stable
- Mean value theorem can be used to generalize conclusions for arbitrary amplitude
  - R. Davidson proof

// Example Applications of Kinetic Stability Theorem

KV Equilibrium:



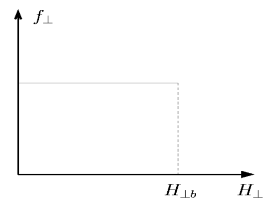
$$f_\perp(H_\perp) = \frac{\hat{n}}{2\pi}\delta(H_\perp - H_{\perp b})$$

$\partial f_\perp / \partial H_\perp$ changes sign

inconclusive stability by theorem

- Full normal mode analysis in Kinetic theory shows strong instabilities when space-charge becomes strong
- Not surprising, delta function represents a highly inverted population in phase-space with "free-energy" to drive instabilities

Waterbag Equilibrium:

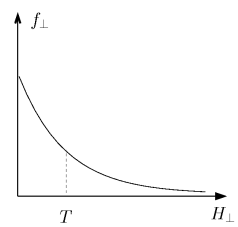


$$f_\perp(H_\perp) = f_0 \Theta(H_{\perp b} - H_\perp)$$

$$\partial f_\perp / \partial H_\perp = f_0 \delta(H_\perp - H_{\perp b}) \leq 0$$

monotonic decreasing, stable by theorem

Thermal Equilibrium:



$$f_\perp(H_\perp) = f_0 \exp(-\beta H_\perp),$$

$$\partial f_\perp / \partial H_\perp = -f_0 \exp(-\beta H_\perp) \leq 0$$

monotonic decreasing, stable by theorem //

S7: rms Emittance Growth and Nonlinear Forces

Fundamental theme of beam physics is to minimize statistical beam emittance growth in transport to preserve focusability on target

Return to the full transverse beam model with:

$$x'' + \kappa_x x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial x} + \text{Applied Nonlinear Field Terms}$$

and express as:

$$\boxed{\begin{aligned} &x''(s) + \kappa_x(s)x(s) = f_x^L(s)x(s) + F_x^{NL}(x,y,s) \\ &f_x^L(s) = \text{Linear Space-Charge Coefficient} \\ &F_x^{NL}(x,y,s) = \text{Nonlinear Forces + Linear Skew Coupled Forces (Applied and Space-Charge)} \end{aligned}}$$

// Examples:

$$f_x^L(s) = \frac{Q}{r_b(s)}$$

Self-field forces within an axisymmetric (mismatched) KV beam core in a continuous focusing model

$$F_x^{NL}(x,y,s) = \text{Im}\left[\underline{b}_3\left(\frac{x+iy}{r_p}\right)^2\right]$$

Electric (with normal and skew components) sextupole optic based on multipole expansions (see lectures on *Particle Equations of Motion*) //

From the definition of the statistical (rms) emittance:

$$\varepsilon_x \equiv 4[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2]^{1/2}$$

Differentiate the squared emittance and apply the chain rule:

$$\begin{aligned}\frac{d}{ds}\varepsilon_x^2 &\equiv 32[\langle xx'\rangle_\perp \langle x'^2\rangle_\perp + \langle x^2\rangle_\perp \langle x'x''\rangle_\perp - \langle xx'\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp \langle xx''\rangle_\perp] \\ &= 32[\langle x^2\rangle_\perp \langle x'x''\rangle_\perp - \langle xx'\rangle_\perp \langle xx''\rangle_\perp]\end{aligned}$$

Insert the equations of motion:

$$x'' + \kappa_x x = f_x^L x + F_x^{NL}$$

The linear terms cancel to show *for any beam distribution* that:

$$\boxed{\frac{d}{ds}\varepsilon_x^2 = 32\left[\langle x^2\rangle_\perp \langle x' F_x^{NL}\rangle_\perp - \langle xx'\rangle_\perp \langle x F_x^{NL}\rangle_\perp\right]}$$

Implications of:

$$\boxed{\frac{d}{ds}\varepsilon_x^2 = 32\left[\langle x^2\rangle_\perp \langle x' F_x^{NL}\rangle_\perp - \langle xx'\rangle_\perp \langle x F_x^{NL}\rangle_\perp\right]}$$

- Emittance evolution/growth is driven by nonlinear or skew coupling forces
  - Nonlinear terms can result from applied or space-charge fields
  - More detailed analysis shows that skew coupled forces cause x-y plane transfer oscillations but there is still a 4D quadratic invariant
- Minimize nonlinear forces to preserve emittance and maintain focusability

If the beam is accelerating, the equations of motion become:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} x' + \kappa_x x = f_x^L x + F_x^{NL}$$

and this result can be generalized (see homework problems) using the normalized emittance:

$$\boxed{\begin{aligned}&\varepsilon_{nx} \equiv \gamma_b\beta_b\varepsilon_x \\ &\frac{d}{ds}\varepsilon_{nx}^2 = 32(\gamma_b\beta_b)^2\left[\langle x^2\rangle_\perp \langle x' F_x^{NL}\rangle_\perp - \langle xx'\rangle_\perp \langle x F_x^{NL}\rangle_\perp\right]\end{aligned}}$$

S8: rms Emittance Growth and Nonlinear Space-Charge Forces

[Wangler et. al, IEEE Trans. Nuc. Sci. 32, 2196 (1985), Reiser, *Charged Particle Beams*, (1994)]

In continuous focusing all nonlinear force terms are from space-charge, giving:

$$\boxed{\frac{d}{ds}\varepsilon_x^2 = -\frac{32q}{m\gamma_b^3\beta_b^2c^2}\left[\langle x^2\rangle_\perp\langle x'\frac{\partial\phi}{\partial x}\rangle_\perp - \langle xx'\rangle_\perp\langle x\frac{\partial\phi}{\partial x}\rangle_\perp\right]}$$

For any axisymmetric beam it can be shown that:

$$\langle x\frac{\partial\phi}{\partial x}\rangle_\perp = \frac{1}{2}\langle\frac{\partial\phi}{\partial r}\rangle_\perp = -\frac{\lambda}{8\pi\epsilon_0}$$

$$\langle x'\frac{\partial\phi}{\partial x}\rangle_\perp = \frac{1}{2}\langle r'\frac{\partial\phi}{\partial x}\rangle_\perp = \frac{1}{8\pi\epsilon_0\lambda}\frac{dW}{ds}$$

$$W = \frac{\epsilon_0}{2}\int d^2x\,|\nabla_\perp\phi|^2$$

$=$ self-field energy (per unit axial length)

For any axisymmetric beam it can also be shown that:

$$\langle xx'\rangle_\perp = \frac{1}{2}\langle rr'\rangle_\perp = -\frac{\langle x^2\rangle_\perp}{\lambda^2}\frac{dW_u}{ds}$$

$W_u =$ W for an rms equivalent uniform density beam

These results give (Wangler, Lapostolle):

$$\boxed{\frac{d}{ds}\varepsilon_x^2 = -4Q\langle x^2\rangle_\perp\frac{d}{ds}\left(\frac{W-W_u}{\lambda^2}\right)}$$

$$\boxed{\frac{d}{ds}\varepsilon_x^2 = -8Q\langle x^2\rangle_\perp\frac{d}{ds}\left(\frac{W-W_u}{\lambda^2}\right)}$$

- Applies to both radially bounded and radially infinite systems
- Result does not require an equilibrium for validity – only axisymmetry
- For a beam with s-variation, this result suggests that *only* the (mismatched) KV equilibrium can subsequently evolve with no change in rms emittance
- Result can be partially generalizable [J. Struckmeier and I. Hofmann, Part. Accel. **39**, 219 (1992)] to an unbunched elliptical beam
  - Result may have implications to existence/nonexistence of nonuniform density Vlasov equilibria in periodic focusing channels

If the rms beam radius does not change much in the beam evolution:

$$r_b^2 = 2\langle x^2\rangle_\perp \simeq \text{const}$$

Then the equation can be integrated to show that:

$$\Delta_{fi}(\varepsilon_x^2) = -4Qr_b^2\Delta_{fi}\left(\frac{W-W_u}{\lambda^2}\right)$$

$\Delta_{fi}(\cdots) \equiv$ Final State Value – Initial State Value

S9: Uniform Density Beams and Extreme Energy States

Construct minima of the self-field energy per unit axial length:

$$\boxed{W = \frac{\epsilon_0}{2} \int d^2x_\perp \, |\nabla_\perp \phi|^2}$$

subject to: $\lambda = \text{const}$ ... fixed line-charge

$r_b = \sqrt{2\langle r^2 \rangle_\perp} = \text{const}$ ... fixed rms equivalent beam radius

Using the method of Lagrange multipliers, vary (Helmholtz free energy):

$$F = W - \mu(\lambda/q)\langle r^2 \rangle_\perp = \int d^2x_\perp \, \left\{ \epsilon_0 \frac{|\nabla_\perp \phi|^2}{2} - \mu r^2 n \right\} \qquad \mu = \text{const}$$

and require that variations satisfy the Poisson equation and conserve charge

$$\nabla_\perp^2 \delta\phi = -\frac{q}{\epsilon_0}\delta n \qquad \delta\phi|_{\text{boundary}} = 0 \qquad \int d^2x_\perp \, \delta n = 0$$

Then variations terminate at 2nd order giving:

$$\delta F = -\int d^2x_\perp \, \{\mu r^2 + \text{const}\} \, \delta n + \epsilon_0 \int d^2x_\perp \nabla_\perp \phi \cdot \nabla_\perp \delta\phi + \frac{\epsilon_0}{2} \int d^2x_\perp |\nabla_\perp \delta\phi|^2$$

Integrating the 2nd term by parts and employing the Poisson equation then gives:

$$\boxed{\delta F = \int d^2x_\perp \, \{q\phi - \mu r^2 - \text{const}\} \, \delta n \; + \; \frac{\epsilon_0}{2} \int d^2x_\perp |\nabla_\perp \delta\phi|^2}$$

For an extremum, the first order term must vanish, giving within the beam:

$$q\phi = \mu r^2 + \text{const}$$

From the Poisson equation, this can only be consistent with a uniform density axisymmetric beam. The 2nd order term is positive definite, immediately implying that this extremum is a global *minimum* of F

Result:

At fixed line charge and rms radius, a uniform density beam minimizes the electrostatic self-field energy

At fixed line charge and rms radius, a uniform density beam minimizes the electrostatic self-field energy

This result, combined with Wangler's Theorem:

$$\boxed{\frac{d}{ds}\varepsilon_x^2 = -Q\langle x^2\rangle_\perp \frac{d}{ds}\left(\frac{W - W_u}{\lambda^2}\right)}$$

shows that:

- Self-field energy drives emittance evolution
  - Nonuniform density => more uniform density <=> local emittance growth
  - Uniform density => more nonuniform density <=> local emittance reduction
- Try to maintain density uniformity to preserve beam emittance
- Results can be partially generalized to 2D elliptical beams

  [J. Struckmeier and I. Hofmann, Part Accel. **39**, 219 (1992)]

S10: Collective Relaxation and rms Emittance Growth

The space-charge profile of intense beams can be born highly nonuniform out of nonideal (real) injectors or become nonuniform due to a variety of (error) processes. Also, low-order envelope matching of the beam may be incorrect due to focusing and/or distribution errors.

How much emittance growth and changes in other characteristic parameters may be induced by relaxation of characteristic perturbations?

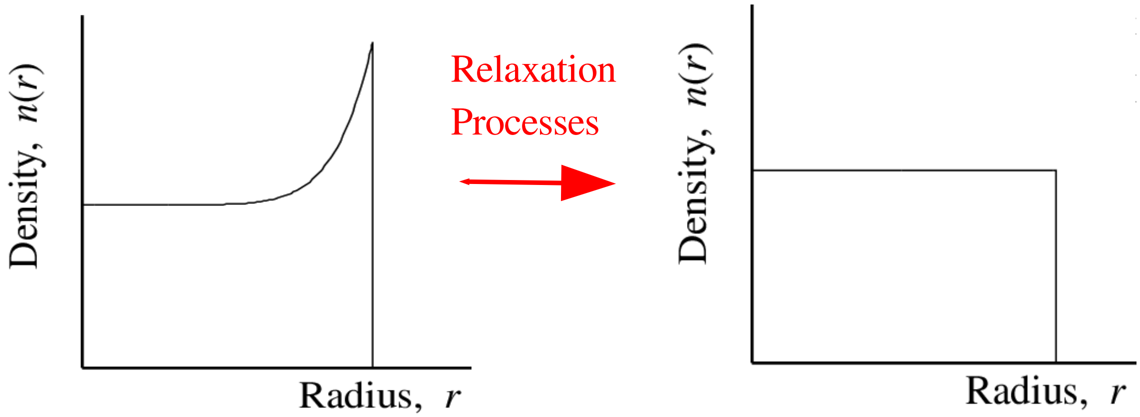
- Employ Global Conservation Constraints of system to bound possible changes
- Assume full relaxation to a final, uniform density state for simplicity

What is the mechanism for the assumed relaxation?

- Collective modes launched by errors will have a broad spectrum
  - Phase mixing can smooth nonuniformities – mode frequencies incommensurate
- Nonlinear interactions, Landau damping, interaction with external errors, ...
- Certain errors more/less likely to relax:
  - Internal wave perturbations expected to relax due to many interactions
  - Envelope mismatch will not (coherent mode) unless amplitudes are very large producing copious halo and nonlinear interactions

Example: Relaxation of nonlinear space-charge waves

Nonuniform Initial Beam | **Uniform Final Beam**



Reference: High resolution self-consistent PIC simulations shown in class

- Continuous focusing and a more realistic FODO transport lattice
  - Relaxation more complete in real lattice due to a richer frequency spectrum
- Relaxations surprisingly rapid: few undepressed betatron wavelengths

Initial Nonuniform Beam Parameterization

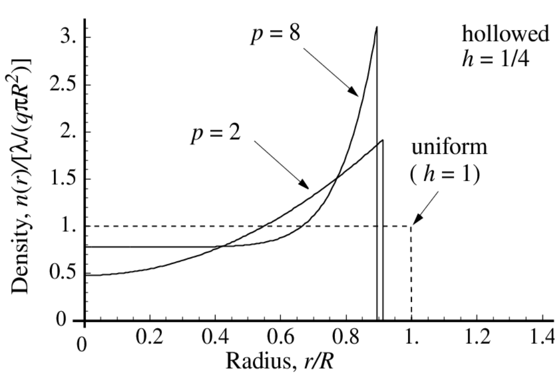
$$n(r) = \begin{cases} \hat{n}\left[1 + \frac{1-h}{h}\left(\frac{r}{r_b}\right)^p\right], & 0 \le r \le r_b \\ 0, & r_b < r \le r_p \end{cases}$$

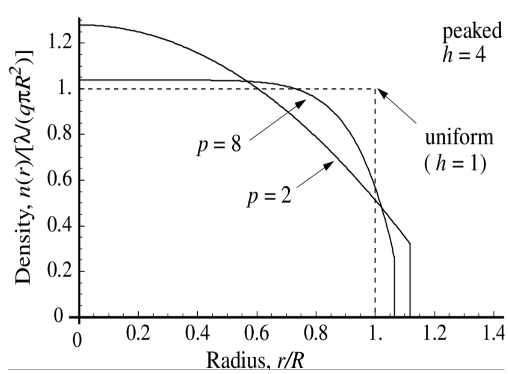
$h =$ hollowing parameter

$p =$ radial power

$$\lambda = \int d^2x_\perp \; n \; = \pi q \hat{n} r_b^2 \left[\frac{(ph+2)}{(p+2)h}\right] \qquad R = 2\langle x^2\rangle_\perp^{1/2} = \sqrt{\frac{(p+2)(ph+4)}{(p+4)(ph+2)}}\, r_b$$

Hollowed Initial Density | **Peaked Initial Density**

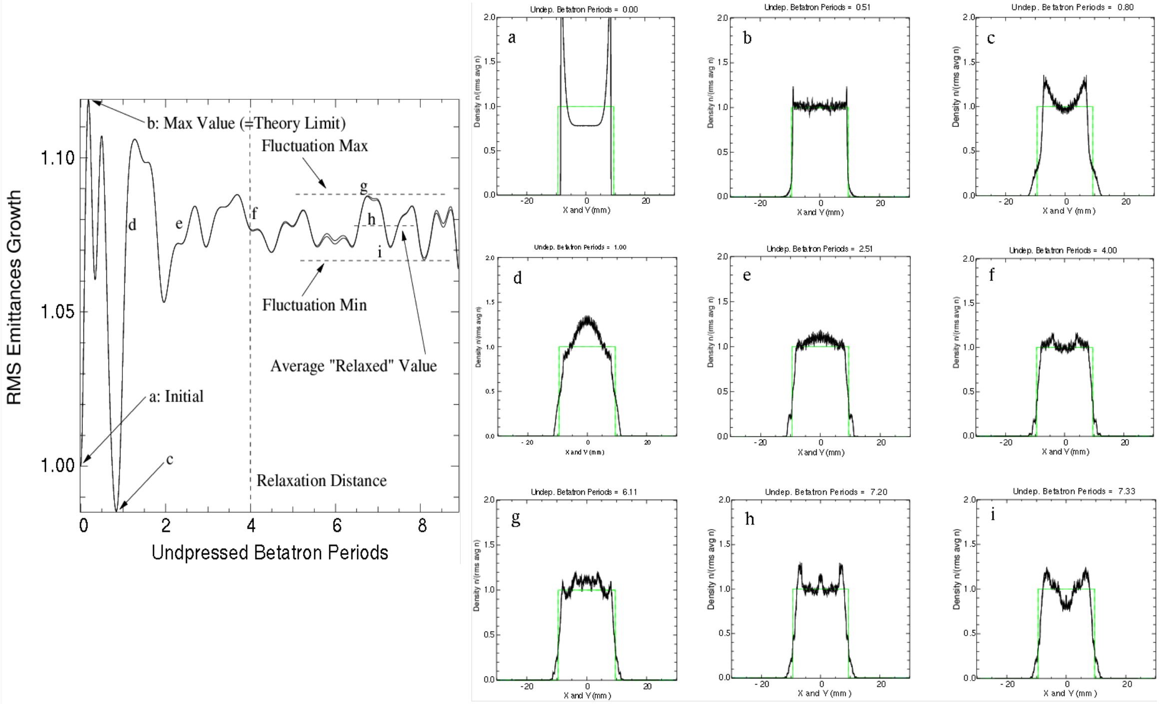




- Analogous definitions are made for the radial temperature profile of the beam

Example Simulations, Initial Nonuniform Beam

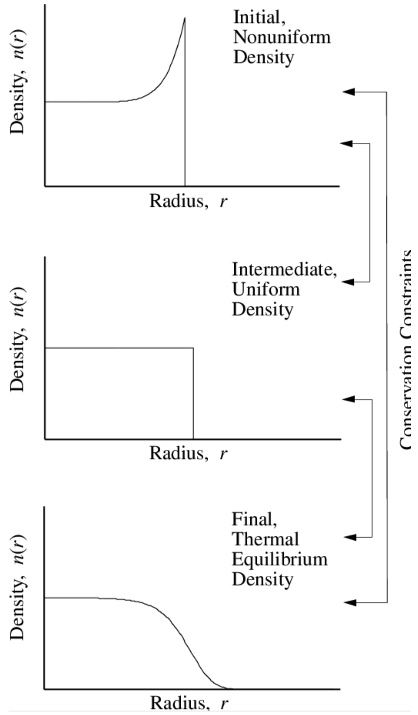
$\sigma/\sigma_0 = 0.2$ Initial density: h=1/4, p=8 Initial Temp: h = infinity, p=2



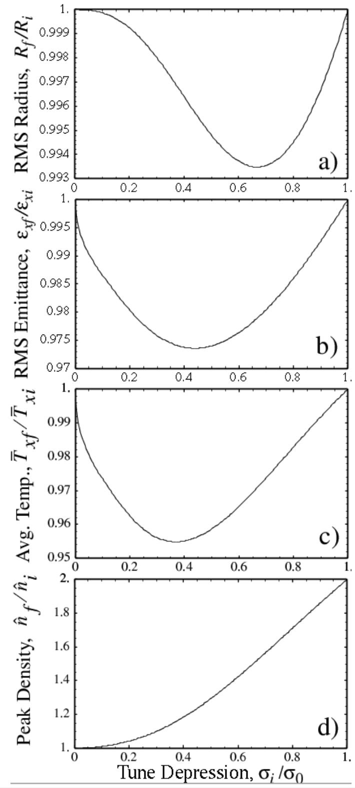
[Lund, Grote, and Davidson, Nuc. Instr. Meth. A 544, 472 (2005)]

Initial beam					Relaxed and transient beam		
σ_i/σ_0	Density		Temperature		Emittance growth		Undep. betatron periods to relax
	h	p	h	p	Theory	Simulation	
0.1	0.25	4	1	arb.	1.57	1.42 (1.57, 1.31–1.52)	3.5
			∞	2		1.45 (1.57, 1.38–1.52)	3.0
			0.5			1.41 (1.57, 1.30–1.52)	3.0
	0.25	8	1	arb.	1.43	1.33 (1.43, 1.28–1.38)	3.5
			∞	2		1.35 (1.43, 1.30–1.40)	4.5
			0.5			1.32 (1.43, 1.26–1.38)	4.0
0.20	0.25	4	1	arb.	1.17	1.11 (1.16, 1.09–1.13)	4.5
			∞	2		1.12 (1.16, 1.10–1.13)	3.0
			0.5			1.11 (1.16, 1.09–1.13)	4.0
	0.25	8	1	arb.	1.12	1.08 (1.12, 1.06–1.09)	5.5
			∞	2		1.08 (1.12, 1.07–1.09)	4.0
			0.5			1.08 (1.12, 1.06–1.09)	4.5

Theory estimates made from global conservation constraints work well but what if the beam relaxed to a smooth thermal equilibrium profile instead?



Essentially no rms changes in 2[nd] step!



Lund, Barnard, and Miller, PAC 1995, p. 3278

S11: Phase Mixing and Landau Damping in Beams

To be covered in future editions of notes

These slides will be corrected and expanded for reference and any future editions of the US Particle Accelerator School class:
Beam Physics with Intense Space Charge, by J.J. Barnard and S.M. Lund

Corrections and suggestions are welcome. Contact:

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